Options. I.

Christopher G. Lamoureux

January 9, 2013

Organizing Themes

The Black-Scholes-Merton option pricing theory is perhaps the most successful model in finance.

This may well be because it takes two fundamental market prices as given: the stock price and the risk-free rate.

This approach of asking what are the restrictions of the absence of arbitrage on *relative* prices is even more popular on Wall Street than in academia.

We now understand that any financial asset's price is an expectation in the equivalent risk-neutral measure. This implies that we need two sets of skills.

- Changing Measures.
- Evaluating stochastic Integrals.

$E\left[\max(S - X, 0)\right]$

Where:

$$dS = \mu S dt + \sigma S dz$$

It follows that $f(S_t|S_s)$ is lognormal.
There are many ways to solve for the value of an option:

- Form a risk-neutral portfolio, identify its sde and solve (What Black and Scholes did).
- Do a change of measure and take expectations.
- Apply Feynman-Kac Theorem.

In any case, we have to understand how to take this slide's eponymous expectation.

Itô's Lemma

Let G be a function of the random variable x.

$$\Delta G \approx \frac{dG}{dx} \Delta x$$

Taylor Series:

$$\Delta G = \frac{dG}{dx} \Delta x + \frac{1}{2} \frac{d^2 G}{dx^2} \Delta x^2 + \frac{1}{6} \frac{d^3 G}{dx^3} \Delta x^3 + \dots$$

Now let G be a function of the random variable x and y:

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 + \dots$$

Itô

So now, consider:

$$dx = a(x,t)dt + b(x,t)dz$$

Then:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x \Delta t \ldots$$
$$\ldots + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots$$

Itô

As Δx and Δy approach 0:

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

We can discretize the Itô process:

$$\Delta x = a\Delta t + b\epsilon\sqrt{\delta t}$$

Itô

The main intuition from Itô's lemma is that the variance of a Wiener process increases at the rate t. So, this means that the expansion:

$$\Delta x^2 = b^2 \epsilon^2 \Delta t + \dots$$

We know $E(\epsilon^2) = 1$ (Why?)
Itô's Lemma:

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} dx^2$$

So for dx an Itô process:

$$dG = \left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

Stock Process

Consider that the stock price, S, follows a geometric Brownian motion:

$$dS = S\mu dt + S\sigma dz$$

Let $G = \ln S$, then:
since: $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$.

$$dG = \frac{1}{S}\left(\mu S dt + \sigma S dz\right) - \frac{\sigma^2 S^2}{2S^2} dt$$

So:

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T \ , \ \sigma^2 T\right]$$

and:

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T \ , \ \sigma^2 T\right]$$

Back to: $E[\max(S - X, 0)]$

$$E\left[\max\left(V - K, 0\right)\right] = \int_K^\infty (V - K) g(V) dV \qquad (1)$$

We know that V is lognormally distributed, so that the mean of $\ln V$ is m:

$$m = \ln\left[E(V)\right] - \frac{\sigma^2}{2}$$

Now let $z = \frac{\ln V - m}{\sigma}$, so:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Back to: $E[\max(S-X,0)]$

So do a change of variable within the integral:

$$E\left[\max\left(V-K,0\right)\right]=\int_{\frac{\ln X-m}{\sigma}}^{\infty}\left(e^{\sigma z+m}-X\right)f(z)dz$$

$$=\int_{\frac{\ln X-m}{\sigma}}^{\infty}e^{\sigma z+m}f(z)dz-X\int_{\frac{\ln X-m}{\sigma}}^{\infty}f(z)dz$$

And:

$$e^{\sigma z+m}f(z)=\frac{1}{\sqrt{2\pi}}e^{\left(-z^2+2\sigma z+2m\right)/2}$$

$$=\frac{1}{\sqrt{2\pi}}e^{\left((-z-\sigma)^2+2m+\sigma^2\right)/2}$$

$$=\frac{e^{m+\frac{\sigma^2}{2}}}{\sqrt{2\pi}}e^{\frac{-(z-\sigma)^2}{2}}$$

$E[\max(S - X, 0)]$ (Continued)

$$E\left[\max\left(V-K,0\right)\right] = e^{m+\frac{\sigma^2}{2}} \cdot f\left(z-\sigma\right)$$

$$= e^{m+\frac{\sigma^2}{2}} \int_{\frac{\ln X - m}{\sigma}}^{\infty} f\left(z-\sigma\right) dz - X \int_{\frac{\ln X - m}{\sigma}}^{\infty} f\left(z\right) dz$$

And:

$$\int_{\frac{\ln X - m}{\sigma}}^{\infty} f\left(z-\sigma\right) dz = 1 - \mathcal{N}\left\{\frac{\ln X - m}{\sigma} - \sigma\right\}$$

(Why?) Alternately:

$$\int_{\frac{\ln X - m}{\sigma}}^{\infty} f\left(z-\sigma\right) dz = \mathcal{N}\left\{\frac{m - \ln X}{\sigma} + \sigma\right\}$$

or

$$\int_{\frac{\ln X - m}{\sigma}}^{\infty} f\left(z-\sigma\right) dz = \mathcal{N}\left\{\frac{\ln\left[\frac{E(V)}{X}\right] + \frac{\sigma^2}{2}}{\sigma}\right\}$$

$E[\max(S - X, 0)]$ (Continued)

Now similarly the second integral:

$$\int_{\frac{\ln X - m}{\sigma}}^{\infty} f(z - \sigma) dz = \mathcal{N}\left\{ \frac{\ln\left[\frac{E(V)}{X}\right] - \frac{\sigma^2}{2}}{\sigma} \right\}$$

Black and Scholes

Solving for the expectation is not the contribution of Black and Scholes and Merton. The key to option pricing (and finance) is ascertaining $E(V)$ and σ in the two normal distributions obtained in the preceding slides.
Black and Scholes set up a hedge portfolio that entails continual rebalancing the stock and option to replicate a riskless asset.
Consider a position that is long Δ shares of the underlying stock and short one call option.
The law of motion for this position is:

$$-\frac{\partial C}{\partial t}dt - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt$$

(Since the stock and otion share the same Brownian motion and the position in the stock cancels this term out of the hedge portfolio.)

Black and Scholes (Continued)

Because this position entails investing $C - \frac{\partial C}{\partial S} S$, it must be that:

$$-\frac{\partial C}{\partial t} dt - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dt = \left(C - \frac{\partial C}{\partial S} S \right) dt \cdot r$$

where r is the instantaneous risk-free rate.
This sets the stage for the famous Black and Scholes stochastic differential equation:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$