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Handout for Class 4

Fall 2015
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Notes for the Equity Premium Puzzle.

Mehra and Prescott specify a standard economy with a representative agent with time-separable utility, where, as we have seen:

$$P_t = E_t \left\{ \sum_{s=t+1}^{\infty} \beta^{s-t} U^{'}(y_s) d_s / U^{'}(y_t) \right\}$$

This economy is sometimes called a Lucas Tree economy because the payoff from the asset is distributed as a dividend (fruit) in each period.

Our demonstration of this model will use a 2-state Markov process, so fundamentally this will have a lot in common with our binomial tree work.

Uncertainty is introduced by the assumption that:

$$y_{t+1} = x_{t+1} y_t \tag{1}$$

and, $x_{t+1} \in \{\lambda_1, \ldots, \lambda_n\}$; a Markov chain with

$$\mathrm{Pr}\{\mathrm{x_{t+1}} = \lambda_{\mathrm{j}}; \mathrm{x_t} = \lambda_{\mathrm{i}}\} = \phi_{\mathrm{ij}}. \tag{2}$$

Furthermore, the dividend represents the entirety of consumption each period (the idea of a "Lucas Tree"). So with CRRA utility,

$$U(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha} \tag{3}$$

we have:

$$P_t = E\left\{\sum_{s=t+1}^{\infty} \beta^{s-t}\frac{y_t^{\alpha}}{y_s^{\alpha}} y_s | x_t, y_t\right\} \tag{4}$$

Mehra and Prescott note that since x is an ergodic process, x_t and y_t characterize the state of the economy, and the value of the risky asset is homogeneous of degree one in y_t. Redefine the state to be (c, i) when $y_t = c$ and $x_t = \lambda_i$.

They show:

$$p(c, i) = w_i c \tag{5}$$

where:

$$w_i = \beta \sum_{j=1}^{n} \phi_{ij} \lambda_j^{1-\alpha}(w_j + 1) \quad i = 1, \ldots, n. \tag{6}$$

This is a system of n equations in n unknowns.

Returns:

$$\begin{aligned} r_{ij} &= \frac{p(\lambda_j c, j) + \lambda_j c - p(c, i)}{p(c, i)} && (7) \\ &= \frac{\lambda_j (w_j + 1)}{w_i} - 1 && (8) \\ & && (9) \end{aligned}$$

So the expected return in state i is:

$$E(r_i) = \sum_{j=1}^{n} \phi_{ij} r_{ij} \tag{10}$$

In this setting the price of the riskless asset is also a function of the state:

$$\begin{aligned} p_i^f &= \beta \sum_{j=1}^{n} \phi_{ij} U^{'}(\lambda_j c)/U^{'}(c) && (11) \\ &= \beta \sum_{j=1}^{n} \phi_{ij} x_j^{-\alpha} && (12) \\ & && (13) \end{aligned}$$

And the return on this riskless asset is: $R_i^f = \frac{1}{p_i^f} - 1$.

Spreadsheet structure:

The Bellman equation:

$$V(w_t, x_t, y_t) = \max_{s_t, B_t} \left\{ U(c) + \beta \sum_{x_{t+1}} V(w_{t+1}, x_{t+1}, y_{t+1}) \phi(x_{t+1}, x_t) \right\} \tag{14}$$

s.t.: $c_t + p(x,y)s_t + q(x,y)B \leq w$.

FOC:

$$U^{'}(c)p(x_t, y_t) = \beta \sum_{x_{t+1}} \frac{\delta V(w_{t+1}, x_{t+1}, y_{t+1})}{\delta w_{t+1}} [p(x_{t+1}, x_{t+1} y_t] \phi(x_{t+1}, x_t) \tag{15}$$

With envelope condition:

$$\frac{\delta V(w_t, x_t, y_t)}{\delta w} = U^{'}(c_t) \tag{16}$$

This results in a familiar restriction on asset prices:

$$p(x_t, y_t) = \beta \sum_{x_{t+1}} \frac{U^{'}(x_{t+1} y_t)}{U^{'}(y_t)} \left[p(x_{t+1}, x_{t+1} y_t) + x_{t+1} y_t \right] \phi(x_{t+1}, x_t) \tag{17}$$

With our CRRA utility function we have:

$$\beta \frac{U^{'}(y_{t+1})}{U^{'}(y_t)} = \beta \left(\frac{y_{t+1}}{y_t} \right)^{-\alpha} = \beta (x_{t+1})^{-\alpha}$$

So we can substitute this into (17), and also impose (guess) the form: $p(x_{i,t}, y_t) = p_i \cdot y$. This gives us the system of equations that is solved (for p) in the accompanying spreadsheet.

Discussion of appropriate values for α. My own results suggest it should lie between 5 and 8.

Campbell (2000, p. 1533) refers to this as an exact solution approach (which he notes is also used by Kandel and Stambaugh 1991 JME).

Barro (2006 QJE) refers back to Reitz' (1988 *JME*) idea that (possibly unobserved) large disasters can explain the puzzle. The problem is that consumption in the US is very smooth. So equity – which has to bear a risk premium since it is positively correlated with consumption – doesn't need a high risk premium.

This is of course partially connected to the idea that there is a selection bias in studying the US economy (i.e., it is not representative).

Gabaix (p. 648) references Barro (2009 *AER*) in identifying an anomalous feature of the CRRA utility– as disaster probability rises, risk premia increase but the risk-free rate decreases so much that asset prices increase. Epstein-Zin preferences break the link between savings behavior and risk-aversion.